Unitarization of uniformly bounded subgroups in finite von Neumann algebras

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Abstract

This note will present a new proof of the fact that every uniformly bounded group of invertible elements in a finite von Neumann algebra is similar to a unitary group. The proof involves metric geometric arguments in the non-positively curved space of positive invertible operators of the algebra; in 1974 Vasilescu and Zsido proved this result using the Ryll-Nardzewsky fixed point theorem.

1 Geometry of the cone of positive operators in a finite algebra

The metric geometry of the cone of positive invertible operators in a finite von Neumann algebra was studied in [1, 5]. In this subsection we recall some facts from these papers.
Let $\mathcal{A}$ be a von Neumann algebra with a finite (normal, faithful) trace τ. Denote by $\mathcal{A}_h$ the set of selfadjoint elements of $\mathcal{A}$, by $G_{\mathcal{A}}$ the group of invertible elements, by $U_{\mathcal{A}}$ the group of unitary operators, and by $\mathcal{P}$ the set of positive invertible operators

$$\mathcal{P} = e^{\mathcal{A}_h} = \{a \in G_{\mathcal{A}} : a > 0\};$$

$\mathcal{P}$ is an open subset of $\mathcal{A}_h$ in the norm topology. Therefore if one regards it as a manifold, its tangent spaces identify with $\mathcal{A}_h$ endowed with the uniform norm $\|\cdot\|$.
We make of $\mathcal{P}$ a weak Banach-Finsler manifold by assigning for each $a \in \mathcal{P}$ the following 2-norm to the tangent space $T_a(\mathcal{P}) \simeq \mathcal{A}_h$

$$\|x\|_{a,2} = \|a^{-\frac{1}{2}} x a^{-\frac{1}{2}}\|_2, \qquad \text{for } x \in \mathcal{A}_h \simeq T_a(\mathcal{P})$$

where

$$\|x\|_2 = \tau(x^2)^{\frac{1}{2}} \qquad \text{for } x \in \mathcal{A}_h.$$

One obtains a geodesic distance d_2 on $\mathcal{P}$ by considering

$$d_2(a,b) = inf\{Lenght(\gamma) : \gamma \text{ is a piecewise smooth curve joining } a \text{ and } b\},$$

where smooth means differentiable in the norm induced topology and the lenght of a curve $\gamma : [0,1] \to \mathcal{P}$ is measured using the norm above:

$$Lenght(\gamma) = \int_0^1 \|\dot{\gamma}(t)\|_{\gamma(t),2} dt.$$

If $\mathcal{A}$ is finite dimensional, i.e. a sum of matrix spaces, this metric is well-known: it is the non positively curved Riemannian metric on the set of positive definite matrices [8].

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If $\mathcal{A}$ is of type II_1, the trace inner product is not complete, so that $\mathcal{P}$ is not a Hilbert-Riemann manifold and $(\mathcal{P}, d_2)$ is not a complete metric space, see Remark 3.21 in [5].
The following holds

- By [1, Th. 3.1 and Th. 3.2] the unique geodesic between a and b for $a, b \in \mathcal{P}$ is given by

$$\gamma_{a,b}(t) = a^{\frac{1}{2}}(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})^t a^{\frac{1}{2}}$$

and has lenght equal to

$$d_2(a,b) := Lenght(\gamma_{a,b}) = \|ln(a^{-\frac{1}{2}}ba^{-\frac{1}{2}})\|_2.$$

- The action of $G_{\mathcal{A}}$ on $\mathcal{P}$ given by $I_g(a) = gag^*$ is isometric, i.e. $d_2(I_g(a), I_g(b)) = d_2(a,b)$, and sends geodesic segments to geodesic segments, i.e. $I_g \circ \gamma_{a,b} = \gamma_{I_g(a), I_g(b)}$ for all $a, b \in \mathcal{P}$ and $g \in G_{\mathcal{A}}$. See the Introduction of [1].
- Let $a \in \mathcal{P}$ and $\gamma : [0,1] \to \mathcal{P}$ be a geodesic. Then [5, Theorem 4.4]

$$d_2(\gamma_0, \gamma_1)^2 + 4d_2(a, \gamma_{\frac{1}{2}})^2 \leq 2(d_2(a,\gamma_0)^2 + d_2(a,\gamma_1)^2)$$

so the metric space $(\mathcal{P}, d_2)$ satisfies the semi-parallelogram law (see Definition 2.1 below).
- By [1, Cor. 3.4] the distance along two geodesics is convex, i.e. $t \mapsto d_2(\gamma_{a_1,b_1}(t), \gamma_{a_2,b_2}(t))$, $[0,1] \to [0, +\infty)$ is convex for $a_1, b_1, a_2, b_2 \in \mathcal{P}$. This implies

$$\begin{aligned} d_2(\gamma_{a_1,b_1}(t), \gamma_{a_2,b_2}(t)) &\leq td_2(\gamma_{a_1,b_1}(0), \gamma_{a_2,b_2}(0)) + (1-t)d_2(\gamma_{a_1,b_1}(1), \gamma_{a_2,b_2}(1)) = \\ &= td_2(a_1,a_2) + (1-t)d_2(b_1,b_2). \end{aligned}$$

If $t_0 \in [0,1]$ is fixed, the continuity of $(a,b) \mapsto \gamma_{a,b}(t_0)$, $\mathcal{P} \times \mathcal{P} \to \mathcal{P}$ in the d_2 metric follows from the above inequality.
- Let $\mathcal{P}_{c_1,c_2} := \{a \in \mathcal{P} : c_1 1 \leq a \leq c_2 1\}$ for $0 < c_1 < c_2$. In $\mathcal{P}_{c_1,c_2}$ the linear metric and the rectifiable distance are equivalent [5, Prop. 3.2], i.e. there are $C > 0$, $C' > 0$ such that

$$\|a-b\|_2 \leq Cd_2(a,b), \qquad d_2(a,b) \leq C'\|a-b\|_2 \qquad a,b \in \mathcal{P}_{c_1,c_2}$$

Since $\|\cdot\|_2$ is complete on subsets of $\mathcal{A}$ which are closed and bounded in the uniform norm and $\mathcal{P}_{c_1,c_2}$ is closed and bounded in the uniform norm $(\mathcal{P}_{c_1,c_2}, d_2)$ is a complete metric space. Also, for $a, b \in \mathcal{P}_{c_1,c_2}$

$$d_2(a,b) \leq C'\|a-b\|_2 \leq C'\|a-b\| \leq 2C'c_2$$

so that $\mathcal{P}_{c_1,c_2}$ is bounded in the d_2 metric.
- $\mathcal{P}_{c_1,c_2}$ is geodesically convex: if $a, b \in \mathcal{P}_{c_1,c_2}$ then $\gamma_{a,b}(t) \in \mathcal{P}_{c_1,c_2}$ for every $t \in [0,1]$, see [2].

2 Non-negatively curved metric spaces

In this subsection we recall some well-known results from metric geometry. A general reference is [4]. For the convenience of the the reader we include the proof of the Bruhat-Tits fixed point theorem.

Definition 2.1. *A metric space (X,d) satisfies the semi-parallelogram law if for all $x, y \in X$ there is a $z \in X$ such that for all $w \in X$ the following inequality holds*

$$d(x,y)^2 + 4d(w,z)^2 \leq 2[d(x,z)^2 + d(y,z)^2].$$

A Bruhat-Tits space is a complete metric space in which the semi-parallelogram law holds.

Remark 2.2. *The point z satisfying this inequality is unique and is called the midpoint between x and y and we denote it by $m(x,y)$. We therefore have a function $m : X \times X \to X$ called the midpoint map.*

Lemma 2.3. Serre's Lemma [7, Ch. XI, Lemma 3.1]
Let (X,d) be a Bruhat-Tits and S a bounded subset of X. Then there is a unique closed ball $B_r[y]$ of minimal radius containing S.

Definition 2.4. *The center y of the closed ball $B_r[y]$ in the previous lemma is called the circumcenter of the bounded set S.*

Theorem 2.5. Bruhat-Tits fixed point theorem [3]
If (X,d) is a Bruhat-Tits space and $I : G \to Isom(X)$ is an action of a group G on X by isometries which has a bounded orbit, then the circumcenter of each orbit is a fixed point of the action.

Proof. We denote the action by $g \cdot x$ for $g \in G$ and $x \in X$. Since the action is isometric and there is a bounded orbit all orbits are bounded. For $x \in X$ let $B_r[y]$ be the unique closed ball of minimal radius which contains $G \cdot x$. If $g \in G$ then $G \cdot x = g \cdot (G \cdot x) \subseteq g \cdot B_r[y] = B_r[g \cdot y]$ where the last equality follows since the action is isometric. From the uniqueness of the closed balls of minimal radius containing $G \cdot x$ we conclude that $g \cdot y = y$. Therefore, $g \cdot y = y$ for every $g \in G$ and y is a fixed point of the action.

□

3 Uniformly bounded subgroups

Definition 3.1. *A subset $A \subseteq \mathcal{P}$ is geodesically convex if $\gamma_{a,b}(t) \in A$ for every $a, b \in A$ and $t \in [0,1]$.*

Definition 3.2. *The convex hull of a subset $S \subseteq \mathcal{P}$ is the smallest geodesically convex set containing S and we denote it by $conv(A)$.*
An alternative definition is

$$conv(S) = \bigcup_{n \in \mathbb{N}} X_n$$

where $X_1 = S$, and $X_{n+1} = \{\gamma_{a,b}(t) : a, b \in X_n, t \in [0,1]\}$ inductively for $n \geq 1$.

Lemma 3.3. *If $C \subseteq \mathcal{P}_{c_1,c_2}$ is a geodesically convex subset then its closure $\overline{C}$ in $(\mathcal{P}_{c_1,c_2}, d_2)$ is geodesically convex.*

Proof. If $a, b \in \overline{C}$ and $t \in [0,1]$ let $(a_n)_n, (b_n)_n$ be sequences in C such that $a_n \to a$, $b_n \to b$. $\gamma_{a_n,b_n}(t) \in C$ for all $n \in \mathbb{N}$ and since $(a,b) \mapsto \gamma_{a,b}(t)$ is continuous on $\mathcal{P}_{c_1,c_2} \times \mathcal{P}_{c_1,c_2}$, $\gamma_{a_n,b_n}(t) \to \gamma_{a,b}(t)$. We conclude that $\gamma_{a,b}(t) \in \overline{C}$.

□

Theorem 3.4. *Let $H \subseteq G_{\mathcal{A}}$ be a uniformly bounded subgroup, i.e. $\sup_{h \in H} \|h\| := M < \infty$. Then there is an $s \in \mathcal{P}_{M^{-1},M}$ such that $shs^{-1} \in U_{\mathcal{A}}$ for every $h \in H$.*

Proof. Consider the action $I : H \to Isom(\mathcal{P})$ given by $I_h(a) = hah^*$ for $h \in H$ and $a \in \mathcal{P}$. We denote $h \cdot a := I_h(a)$. Since $H \cdot 1 = \{hh^* : h \in H\} \subseteq \mathcal{P}_{M^{-2},M^2}$ and $\mathcal{P}_{M^{-2},M^2}$ is geodesically convex $conv(H \cdot 1) \subseteq \mathcal{P}_{M^{-2},M^2}$. Also, since $\mathcal{P}_{M^{-2},M^2}$ is closed in $(\mathcal{P}, d_2)$, $\overline{conv}(H \cdot 1) \subseteq \mathcal{P}_{M^{-2},M^2}$.
We adopt the notation of Definition 3.2. $X_1 = H \cdot 1$ is invariant for the action and since the action sends geodesics segments to geodesic segments, if X_n is invariant then X_{n+1} is invariant for all $n \geq 1$. We conclude that $conv(H \cdot 1) = \bigcup_{n \in \mathbb{N}} X_n$ is invariant. Since the action is also isometric $\overline{conv}(H \cdot 1)$ is an invariant subset and we can restrict the action to this subset.
Note that $\overline{conv}(H \cdot 1)$ is a geodesically convex subset of $\mathcal{P}$, in $(\mathcal{P}, d_2)$ the semi-parallelogram holds and the midpoint of $a, b \in \mathcal{P}$ is $\gamma_{a,b}(\frac{1}{2})$, so this law also holds in $(\overline{conv}(H \cdot 1), d_2)$. Since $\overline{conv}(H \cdot 1)$

is a closed subset of the complete metric space $(\mathcal{P}_{M^{-2},M^2}, d_2)$, $(\overline{conv}(H \cdot 1), d_2)$ is a complete metric space. We conclude that $(\overline{conv}(H \cdot 1), d_2)$ is a Bruhat-Tits space.
Since $\mathcal{P}_{M^{-2},M^2}$ is bounded in the d_2 metric $\overline{conv}(H \cdot 1)$ is bounded in this metric. Therefore the action has bounded orbits and the Bruhat-Tits fixed point theorem states that the circumcenter $a \in \overline{conv}(H \cdot 1)$ of $H \cdot 1$ satisfies $I_h(a) = hah^* = a$ for all $h \in H$. Then

$$\begin{aligned} 1 = a^{-\frac{1}{2}} a a^{-\frac{1}{2}} &= a^{-\frac{1}{2}} hah^* a^{-\frac{1}{2}} = (a^{-\frac{1}{2}} h a^{\frac{1}{2}})(a^{\frac{1}{2}} h^* a^{-\frac{1}{2}}) \\ &= (a^{-\frac{1}{2}} h a^{\frac{1}{2}})(a^{-\frac{1}{2}} h a^{\frac{1}{2}})^* \qquad \text{for all } h \in H \end{aligned}$$

so that $a^{-\frac{1}{2}} H a^{\frac{1}{2}} \subseteq U_{\mathcal{A}}$.
Since $a \in \mathcal{P}_{M^{-2},M^2}$, then $a^{\frac{1}{2}} \in \mathcal{P}_{M^{-1},M}$ because the square root is an operator monotone function [6, Prop. 4.2.8]. Taking $s = a^{\frac{1}{2}}$ we get the unitarizer stated in the theorem.

□

Remark 3.5. *The unitarizability of a uniformly bounded subgroup H of the group of bounded linear operators acting on a Hilbert space was obtained independently in the 50s by Day, Dixmier and Nakamura, see [9] and the references therein, assuming that H is amenable. In that context the unitarizer s was obtained as the square root of the center of mass of $\{hh^*\}_{h \in H}$. In the present note the unitarizer is the square root of the circumcenter of that same set; we assume however the existence of a finite trace, and in this setting, Vasilescu and Zsido [10] proved in the 70s the result (without the assumption on amenability) using the Ryll-Nardzewsky fixed point theorem which involves weak topologies.*

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